Eur. Phys. J. C 13, 671–679 (2000)
Digital Object Identifier (DOI) 10.1007/s100520000274

THE EUROPEAN PHYSICAL JOURNAL C

Approximate analytic solutions of the renormalization group equations for Yukawa and soft couplings in SUSY models

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Received: 28 July 1999 / Revised version: 8 October 1999 /
Published online: 17 March 2000 – © Springer-Verlag 2000

Abstract. We present simple analytical formulae which describe solutions of the RG equations for Yukawa couplings in SUSY gauge theories with the accuracy of a few per cent. Performing the Grassmannian expansion in these solutions, one finds those for all the soft couplings and masses. The solutions clearly exhibit the fixed point behaviour which can be calculated analytically. A comparison with numerical solutions is made.

1 Introduction

The renormalization group equations (RGEs) for the rigid couplings and soft parameters in SUSY gauge theories play a crucial role in applications. Actually, all predictions of the MSSM are based on solutions to these equations in leading and next-to-leading orders [1]. Typically, one has three gauge couplings, one or three Yukawa couplings (for the case of low or high $\tan\beta$, respectively) and a set of soft couplings. In leading order, solutions to the RGEs for the gauge couplings are simple; however, already for Yukawa couplings, they are known in an analytical form only for the low $\tan\beta$ case, where only the top coupling is left. Moreover, even in this case solutions for the soft terms look rather cumbersome and difficult to explore [2].

In a recent paper [3], it has been shown that solutions to the RGEs for the soft couplings follow from those for the rigid ones in a straightforward way. [1] One takes the solution for the rigid coupling (gauge or Yukawa), substitute instead of the initial conditions their modified expressions

$$\alpha_i \Rightarrow \tilde{\alpha}_i = \alpha_i(1 + M_i\eta + \bar{M}_i\bar{\eta} + 2M_i\bar{M}_i\eta\bar{\eta}), \quad (1)$$

$$Y_k \Rightarrow \tilde{Y}_k = Y_k(1 - A_k\eta - \bar{A}_k\bar{\eta} + A_k\bar{A}_k\eta\bar{\eta} + \Sigma_k\eta\bar{\eta}). \quad (2)$$

where $\eta = \theta^2, \bar{\eta} = \bar{\theta}^2$, and θ and $\bar{\theta}$ are the Grassmannian parameters, and expand over these parameters. This gives the solution to the RGEs for the soft couplings.

Hereafter the following notation is used:

$$\alpha_i \equiv \frac{g_i^2}{16\pi^2}, \quad Y_k \equiv \frac{y_k^2}{16\pi^2}, \quad \Sigma_k = \sum_{j=1}^{3} m_j^2. \quad (3)$$

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[1] Here we follow the approach advocated in [4]. A similar method which was used in a somewhat different way has been also presented in [5,6].

where g_i and y_k are the gauge and Yukawa couplings, respectively, and m_j^2 are the soft masses associated with each scalar field.

This procedure, however, assumes that one knows solutions to the RGEs for the rigid couplings in the analytic form. For instance, in the case of the MSSM in the low $\tan\beta$ regime this allows one to get solutions for the soft couplings and masses simpler than those known in the literature (see [3]). At the same time, in many cases such solutions are unknown. Actual examples are the MSSM with high $\tan\beta$ and NMSSM. One is bound to solve the RGEs numerically when the number of coupled equations increases dramatically with the soft terms being included.

Below we propose simple analytical formulae which give an approximate solution to the RGEs for Yukawa couplings in an arbitrary SUSY theory with the accuracy of a few per cent. Performing the Grassmannian expansion in these approximate solutions one can get those for the soft couplings in a straightforward way. As an illustration we consider the MSSM in the high $\tan\beta$ regime.

One can immediately see that approximate solutions obtained in this way possess infrared quasi-fixed points [7] which can be found analytically. They appear in the limit when the initial values of the Yukawa couplings are much larger than those for the gauge ones. Then, one can analytically trace how the initial conditions for the soft terms disappear from their solutions in the above mentioned limit.

The paper is organized as follows. In Sect. 2, we consider the MSSM in the low $\tan\beta$ regime, where all solutions are known analytically and describe briefly the Grassmannian expansion. In Sect. 3, we present our approximate solutions for the Yukawa couplings and obtain those for the soft terms. We also present a numerical illustration and compare approximate solutions with the numerical ones. The fixed point behaviour is discussed.

Section 4 contains our conclusions. The explicit formulae for the soft couplings and masses are given in the Appendices.

2 The MSSM: exact solutions in the low $\tan\beta$ case

Consider the MSSM in the low $\tan\beta$ regime. One has three gauge and one Yukawa coupling. The one-loop RG equations are

$$\dot{\alpha}_i = -b_i\alpha_i^2, \quad b_i = (\frac{33}{5}, 1, -3), \quad i = 1,2,3. \tag{4}$$

$$\dot{Y}_t = Y_t\left(\frac{16}{3}\alpha_3 + 3\alpha_2 + \frac{13}{15}\alpha_1 - 6Y_t\right). \tag{5}$$

with the initial conditions $\alpha_i(0) = \alpha_0$, $Y_t(0) = Y_0$, and $t = \ln(M_X^2/Q^2)$. Their solutions are given by [2]

$$\alpha_i(t) = \frac{\alpha_0}{1+b_i\alpha_0 t}, \quad Y_t(t) = \frac{Y_0 E(t)}{1+6Y_0F(t)}, \tag{6}$$

where

$$E(t) = \prod_i (1+b_i\alpha_0 t)^{c_i/b_i}, \quad c_i = \left(\frac{13}{15}, 3, \frac{16}{3}\right), \tag{7}$$

$$F(t) = \int_0^t E(t')\mathrm{d}t'. \tag{8}$$

To get solutions for the soft terms, it is enough to perform the substitution $\alpha \to \tilde{\alpha}$ and $Y \to \tilde{Y}$ for the initial conditions in (6) and expand over η and $\bar{\eta}$. Expanding the gauge coupling in (6) up to η one has (hereafter we assume $M_{i0} = m_{1/2}$)

$$M_i(t) = \frac{m_{1/2}}{1+b_i\alpha_0 t}. \tag{9}$$

Performing the same expansion for the Yukawa coupling and using the relations

$$\left.\frac{\mathrm{d}\tilde{E}}{\mathrm{d}\eta}\right|_\eta = m_{1/2}t\frac{\mathrm{d}E}{\mathrm{d}t}, \quad \left.\frac{\mathrm{d}\tilde{F}}{\mathrm{d}\eta}\right|_\eta = m_{1/2}(tE-F). \tag{10}$$

one finds the well-known expression [2]

$$A_t(t) = \frac{A_0}{1+6Y_0F} + m_{1/2}\left(\frac{t}{E}\frac{\mathrm{d}E}{\mathrm{d}t} - \frac{6Y_0(tE-F)}{1+6Y_0F}\right). \tag{11}$$

To get the solution for the Σ term, one has to make an expansion over η and $\bar{\eta}$. This can be done with the help of the following relations:

$$\left.\frac{\mathrm{d}^2\tilde{E}}{\mathrm{d}\eta\mathrm{d}\bar{\eta}}\right|_{\eta,\bar{\eta}} = m_{1/2}^2\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2\frac{\mathrm{d}E}{\mathrm{d}t}\right), \quad \left.\frac{\mathrm{d}^2\tilde{F}}{\mathrm{d}\eta d\bar{\eta}}\right|_{\eta,\bar{\eta}} = m_{1/2}^2 t^2\frac{\mathrm{d}E}{\mathrm{d}t}. \tag{12}$$

As a result one has [3]

$$\Sigma_t(t) = \frac{\Sigma_0 - A_0^2}{1+6Y_0F} + \frac{(A_0 - m_{1/2}6Y_0(tE-F))^2}{(1+6Y_0F)^2} + m_{1/2}^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{E}\frac{\mathrm{d}E}{\mathrm{d}t}\right) - \frac{6Y_0}{1+6Y_0F}t^2\frac{\mathrm{d}E}{\mathrm{d}t}\right]. \tag{13}$$

which is much simpler than what one finds in the literature [2], though coinciding with it after some cumbersome algebra.

One can also write down solutions for the individual masses using the Grassmannian expansion of those for the corresponding superfield propagators. For the first two generations one has

$$m_{Q_L}^2 = m_0^2 + \frac{1}{2}m_{1/2}^2\left(\frac{16}{3}f_3 + 3f_2 + \frac{1}{15}f_1\right), \tag{14}$$

$$m_{U_R}^2 = m_0^2 + \frac{1}{2}m_{1/2}^2\left(\frac{16}{3}f_3 + \frac{16}{15}f_1\right), \tag{15}$$

$$m_{D_R}^2 = m_0^2 + \frac{1}{2}m_{1/2}^2\left(\frac{16}{3}f_3 + \frac{4}{15}f_1\right), \tag{16}$$

$$m_{H_1}^2 = m_0^2 + \frac{1}{2}m_{1/2}^2\left(3f_2 + \frac{3}{5}f_1\right), \tag{17}$$

$$m_{L_L}^2 = m_0^2 + \frac{1}{2}m_{1/2}^2\left(3f_2 + \frac{3}{15}f_1\right), \tag{18}$$

$$m_{E_R}^2 = m_0^2 + \frac{1}{2}m_{1/2}^2\left(\frac{12}{5}f_1\right), \tag{19}$$

where

$$f_i = \frac{1}{b_i}\left(1 - \frac{1}{(1+b_i\alpha_0 t)^2}\right). \tag{20}$$

The third generation masses get a contribution from the top Yukawa coupling

$$m_{b_R}^2 = m_{D_R}^2, \tag{21}$$

$$m_{b_L}^2 = m_{D_L}^2 + \Delta/6, \tag{22}$$

$$m_{t_R}^2 = m_{U_R}^2 + \Delta/3, \tag{23}$$

$$m_{t_L}^2 = m_{U_L}^2 + \Delta/6, \tag{24}$$

$$m_{H_2}^2 = m_{H_1}^2 + \Delta/2, \tag{25}$$

where Δ is related to Σ_t (13) by

$$\begin{aligned}\Delta &= \Sigma_t - \Sigma_0 - m_{1/2}^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{E}\frac{\mathrm{d}E}{\mathrm{d}t}\right)\right] \\ &= \frac{\Sigma_0 - A_0^2}{1+6Y_0F} + \frac{(A_0 - m_{1/2}6Y_0(tE-F))^2}{(1+6Y_0F)^2} \\ &\quad - m_{1/2}^2\frac{6Y_0}{1+6Y_0F}t^2\frac{\mathrm{d}E}{\mathrm{d}t} - \Sigma_0.\end{aligned}$$

With analytic solutions (6,11,13) one can analyze the asymptotic and, in particular, find the infrared quasi-fixed points [7] which correspond to $Y_0 \to \infty$

$$Y_t^{\mathrm{FP}} = \frac{E}{6F}, \tag{26}$$

$$A_t^{\mathrm{FP}} = m_{1/2}\left(\frac{t}{E}\frac{\mathrm{d}E}{\mathrm{d}t} - \frac{tE-F}{F}\right), \tag{27}$$

$$\Sigma_t^{\mathrm{FP}} = m_{1/2}^2\left[\left(\frac{tE-F}{F}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{E}\frac{\mathrm{d}E}{\mathrm{d}t}\right) - \frac{t^2}{F}\frac{\mathrm{d}E}{\mathrm{d}t}\right]. \tag{28}$$

One can clearly see that the dependence on Y_0, A_0 and Σ_0 disappears from (26)–(28). Some residual dependence on

m_0^2 is left for the soft masses and partially cancels with that of Δ.

Below we demonstrate how the same procedure works in the case of approximate solutions. As a realistic example we take the MSSM in the high $\tan\beta$ regime.

3 The MSSM: approximate solutions in the high $\tan\beta$ case

The one-loop RGEs for the Yukawa couplings in this case look like

$$\dot{Y}_t = Y_t\left(\frac{16}{3}\alpha_3 + 3\alpha_2 + \frac{13}{15}\alpha_1 - 6Y_t - Y_b\right),$$
$$\dot{Y}_b = Y_b\left(\frac{16}{3}\alpha_3 + 3\alpha_2 + \frac{7}{15}\alpha_1 - Y_t - 6Y_b - Y_\tau\right),$$
$$\dot{Y}_\tau = Y_\tau\left(3\alpha_2 + \frac{9}{5}\alpha_1 - 3Y_b - 4Y_\tau\right).$$

Since the exact solution is absent and might be too cumbersome, we look for an approximate one in a simple form similar to that of (6).

3.1 The choice of the approximate solution

In choosing approximate solutions we follow the idea of [8] where an approximate solution for Y_t and Y_b ignoring Y_τ has been proposed. Our suggestion is to consider separate brackets for each propagator entering into the Yukawa vertex. Then one has the following expressions for the Yukawa couplings:

$$Y_t = \frac{Y_{t0}E_t}{[1 + A(Y_{t0}F_t + Y_{b0}F_b)]^{1/A}[1 + 2BY_{t0}F_t]^{1/B}} \times \frac{1}{[1 + 3CY_{t0}F_t]^{1/C}}, \quad \frac{1}{A} + \frac{1}{B} + \frac{1}{C} = 1$$
$$Y_b = \frac{Y_{b0}E_b}{[1 + A(Y_{t0}F_t + Y_{b0}F_b)]^{1/A}[1 + 2BY_{b0}F_b]^{1/B}} \times \frac{1}{[1 + C(3Y_{b0}F_b + Y_{\tau 0}F_\tau)]^{1/C}},$$
$$Y_\tau = \frac{Y_{\tau 0}E_\tau}{[1 + A'Y_{\tau 0}F_\tau]^{1/A'}[1 + 2B'Y_{\tau 0}F_\tau]^{1/B'}} \times \frac{1}{[1 + C(3Y_{b0}F_b + Y_{\tau 0}F_\tau)]^{1/C}},$$
$$\frac{1}{A'} + \frac{1}{B'} + \frac{1}{C} = 1$$

where the brackets correspond to the Q, U, H_2, Q, D, H_1 and L, E, H_1 propagators, respectively. Here E_t and F_t are given by (7) and (8), and E_b and E_τ have the same form but with $c_i^{(b)} = (7/15, 3, 16/3)$ and $c_i^{(\tau)} = (9/5, 3, 0)$, respectively.

The brackets are organized so that they reproduce the contributions of particular diagrams to the corresponding anomalous dimensions. The coefficients A, B, C, A' and B' are arbitrary and their precise values are not so important. When Yukawa couplings Y_{i0} are small enough, one can make an expansion in each bracket, and the dependence of these coefficients disappears. However, for large couplings, which are of interest for us because of the fixed points, we have some residual dependence. The requirement that the sum of the exponents equals 1 follows from a comparison with the RGEs. Solutions are close to the exact ones when the brackets are roughly equal to each other. Apparently, since $F_\tau < F_t \sim F_b$ and $Y_\tau \le Y_b \le Y_t$ one cannot completely satisfy this requirement. Our choice of the coefficients A, B, C, A' and B' is dictated mainly by simplicity. In the following we choose them as

$$B = A, C = 2/3A \;\rightarrow\; A = 7/2,\; B = 7/2,\; C = 7/3, \quad (29)$$
$$B' = A'/2 \;\rightarrow\; A' = 21/4,\; B' = 21/8. \quad (30)$$

This gives approximate solutions like

$$Y_t \approx \frac{Y_{t0}E_t}{\left[1 + \frac{7}{2}(Y_{t0}F_t + Y_{b0}F_b)\right]^{2/7}[1 + 7Y_{t0}F_t]^{5/7}}, \quad (31)$$
$$Y_b \approx \frac{Y_{b0}E_b}{\left[1 + \frac{7}{2}(Y_{t0}F_t + Y_{b0}F_b)\right]^{2/7}[1 + 7Y_{b0}F_b]^{2/7}} \times \frac{1}{\left[1 + \frac{7}{3}(3Y_{b0}F_b + Y_{\tau 0}F_\tau)\right]^{3/7}}, \quad (32)$$
$$Y_\tau \approx \frac{Y_{\tau 0}E_\tau}{\left[1 + \frac{21}{4}Y_{\tau 0}F_\tau\right]^{4/7}\left[1 + \frac{7}{3}(3Y_{b0}F_b + Y_{\tau 0}F_\tau)\right]^{3/7}}. \quad (33)$$

Solutions for A_i and Σ_i can be obtained by a Grassmannian expansion with the initial conditions

$$A_i(0) = A_0, \quad \Sigma_i(0) = \Sigma_0. \quad (34)$$

These initial conditions correspond to the so-called universality hypothesis which we follow in our numerical illustration for simplicity. However, one can choose arbitrary initial conditions for the soft terms when needed. This leads to an obvious modification of the formulae.

One can also get the corresponding solutions for the individual soft masses. This can be achieved either by a Grassmannian expansion of the corresponding brackets in (31)–(33), or by expressing the masses through the Σs in an exact way. The second way gives a slightly better agreement with the numerical solutions (see below). We present the explicit expressions for the soft terms and masses in Appendix A.

3.2 Numerical analysis

We start by investigating the precision of the approximate solutions for the Yukawa couplings. To estimate the accuracy, we introduce a relative error which is defined as

$$\varepsilon = \frac{Y_{\text{approx}} - Y_{\text{numeric}}}{Y_{\text{numeric}}}, \quad (35)$$

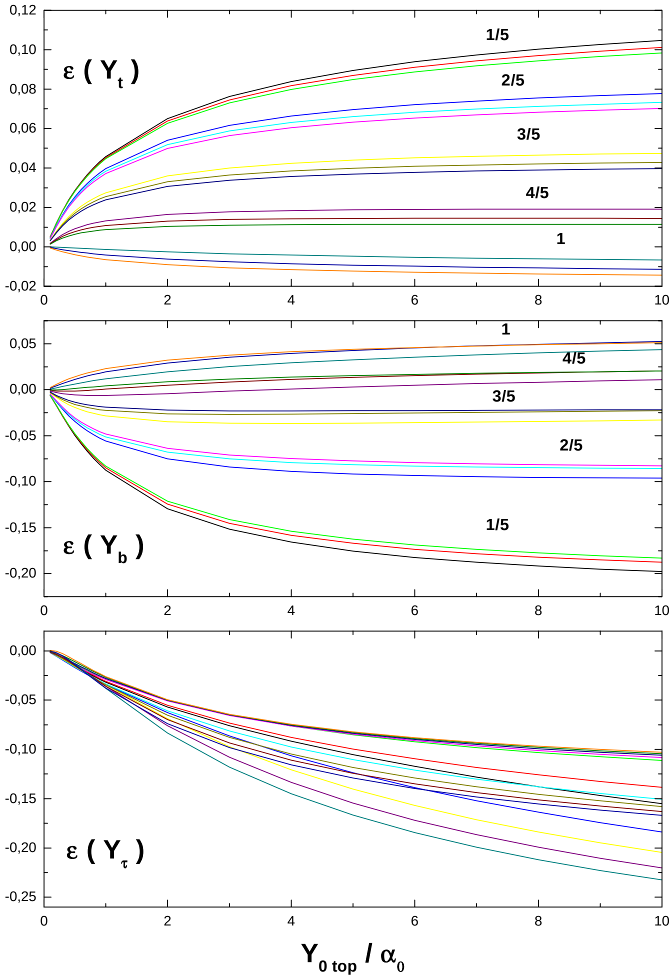

Fig. 1. Y_t, Y_b and Y_τ approximation errors. Numerical labels show the ratio Y_{b0}/Y_{t0} for the corresponding groups of curves (split by different values of $Y_{\tau 0}$)

and which corresponds to the M_Z scale ($t = 66$) at the end of the integration range. The accuracy for the solutions of the soft terms is defined in the same way.

Let us take at the beginning all three Yukawa couplings to be equal at the GUT scale and to have their common value Y_0 in the range $(0.01 \div 25)\alpha_0$. The upper limit is taken in order not to leave the perturbativity regime. We find that for $Y_0 \leq \alpha_0$ the approximation errors are less than 3% for all Ys. While for Y_t it remains smaller than 2% over the whole range of initial values at the GUT scale, for Y_b the error increases up to 4% and for Y_τ up to 14% (for large values of Y_0). It is worth mentioning that for small Y_0 (around $\alpha_0/2$ and below) the accuracy is very good (fractions of a per cent or better).

Consider now $Y_{b0} = Y_{\tau 0} = Y_0 \leq 10\alpha_0$ and let the top Yukawa coupling vary within the limits $1 \leq Y_{t0}/Y_0 \leq 10$ in order to examine the applicability of our formulae. In this case the accuracy is spoilt a little bit with increasing initial values. Namely, the error for Y_t increases up to 10%, and for Y_b and Y_τ up to 20%. However, if one keeps Y_0 in the range $(1/100 \div 1/2)\alpha_0$ the accuracy for Y_t remains better than 3%, and for Y_b and Y_τ better than 10%.

The particular case considered above seems to have the worst accuracy. This is not surprising since our approximate formulae are supposed to work best when all three Yukawa couplings are nearly equal. If we keep $Y_{\tau 0} \leq Y_{b0} \leq Y_{t0}$ and the relative ratios less than 5, we get an average error of less than 5% for Y_t, about 5% for Y_b and 10% for Y_τ. This statement is illustrated in Fig. 1. For each Yukawa coupling we have plotted the error as a function of Y_{t0} in the range $(1/10 \div 10)\alpha_0$. The ratios are kept within the region $1 \leq Y_{t0}/Y_{b0} \leq 5$ and $1 \leq Y_{b0}/Y_{\tau 0} \leq 3$.

Further on, we narrow the range of initial values up to $(1/10 \div 10)\alpha_0$ because the errors (defined as in (35)) come to an asymptotic value for $Y_0 > 10\alpha_0$ and almost vanish for $Y_0 < \alpha_0/10$. The comparison of numerical and approximated solutions is shown in Fig. 2 for three different sets of Y_0s. The approximate solutions follow the numerical ones quite well, preserving their shape, and they have a high accuracy, especially in the case of equal Yukawa couplings at the GUT scale. However, as can be seen from the top of Fig. 2, one can take arbitrary initial conditions for the Yukawa couplings, in particular those which are needed to fit the $t/b/\tau$ masses, and to use our approximate solutions for these purposes.

For the soft couplings, A, we take the initial values at the GUT scale to be $A_0 = (-2, -1, 0, 1, 2)m_{1/2}$ and leave Y_0s in the narrow range as above. Then, we get an accuracy of $(3 \div 5)\%$ for A_t and A_b. For A_τ the approximation is worse when A_0 is taken to be negative or smaller than $m_{1/2}$ (see Fig. 2), but things go better for large initial values of A_0 and we get an accuracy of about 10%. Again it should be mentioned that this is an accuracy at the end point where A_τ itself is close to 0 and the accuracy defined as (35) merely gives an odd hint of the precision. Along the curves the accuracy is much better. In Fig. 2 we have plotted the behaviour of A_t, A_b and A_τ for three different initial values of A_0, namely $\{-m_{1/2}, 0, m_{1/2}\}$ and for one set of Y_0s. As for the Σs, keeping the range of parameter space for Y_0 and A_0 as above, we get an accuracy of typically 2% for Σ_t (even better for fairly equal Y_0s). For Σ_b the precision is around 4%. With Σ_τ we get into the same troubles as for A_τ. The approximation becomes good (about 10%) only for a large enough ratio of $m_0^2/m_{1/2}^2$. The approximation errors for As and Σs are linked with those for Y. If one considers only the sets of small initial values for Y_0 (less than $\alpha_0/2$), then the Σs are approximated with a precision better than 1%, regardless of the A_0 values. The precision for Σ increases with A_0, but this dependence is not so striking as the one on Y_0.

The approximate formulae for the soft masses may be derived from the Σs using (39)–(45). In this case the approximate solutions give an accuracy of about $1 \div 3\%$ for $m^2_{Q_3}$, $m^2_{U_3}$ and $(3 \div 5)\%$ for $m^2_{D_3}$. For the Higgs masses we get a good approximation (of about 5% on average) for $m^2_{H_2}$, and a satisfactory one for $m^2_{H_1}$ (typically 10%). This accuracy is almost insensitive to the A_0 variation (we took it to be in the range $(-2 \div 2)m_{1/2}$) and on the ratio $m_0^2/m_{1/2}^2$ (taken to be $0.5 \div 2$).

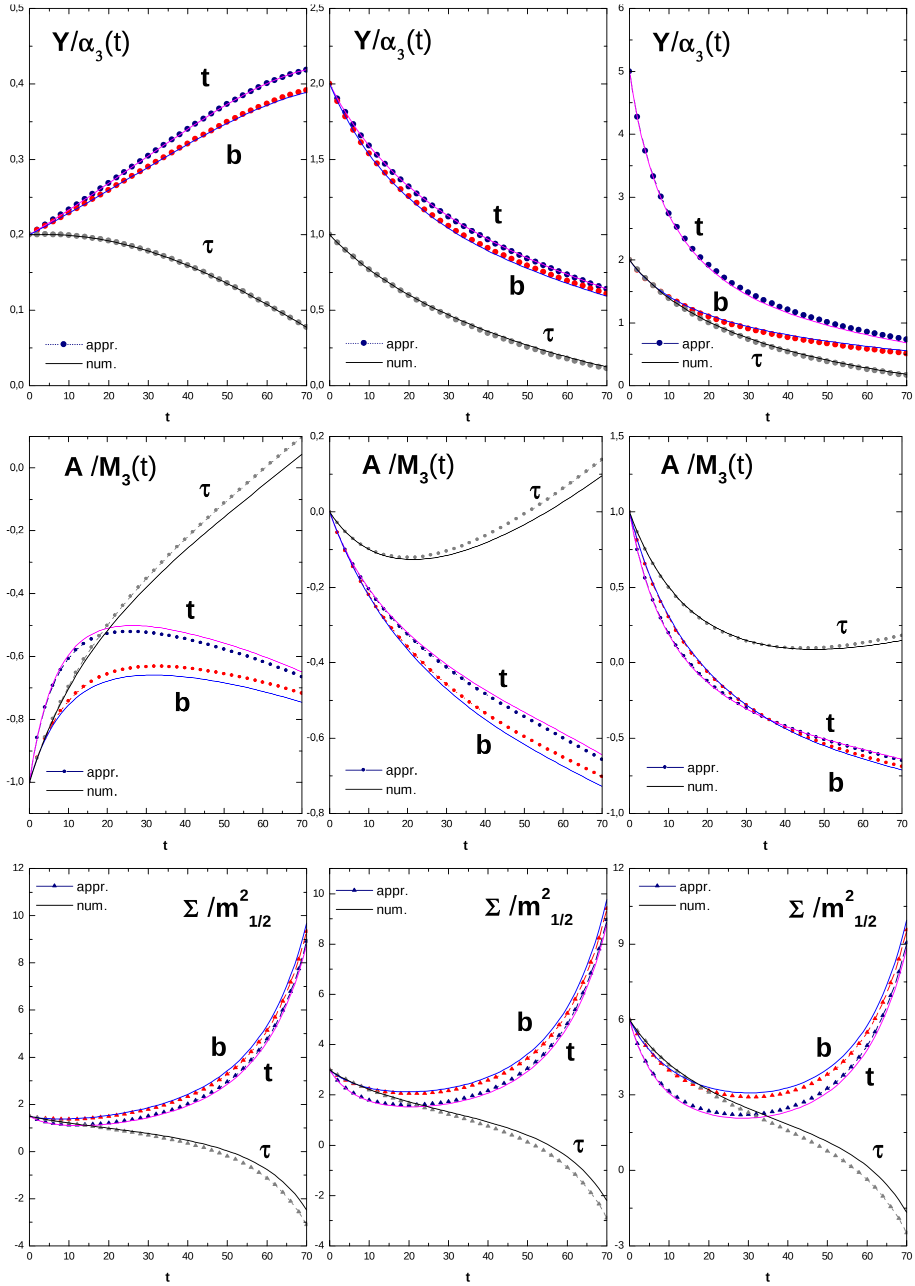

Fig. 2. Comparison of approximate and numerical solutions for Y, A and Σ. The evolution of the soft terms A and Σ has been plotted for $Y_{t0} = 5\alpha_0, Y_{b0} = Y_{\tau 0} = 2\alpha_0$. Dotted lines correspond to the analytical approximation, solid lines to the numerical solution

The slepton masses (see Fig. 3) are not approximated properly in an analogous way. This is mainly due to the less accurate approximation of Y_τ.

As a concluding remark on the numerical analysis, it should be mentioned that one has a rather good approximation for small (less than $\alpha_0/2$) initial values of the Yukawa couplings. For larger values of Ys one has a good approximation especially in the case of unification of the Yukawa couplings.

3.3 The fixed points

One can easily see that the solutions (31)–(33) exhibit the quasi-fixed point behaviour when the initial values $Y_{t0} = Y_{b0} = Y_{\tau 0} \geq \alpha_0$. In this case, one can drop 1 in the denominator and the resulting expressions become independent of the initial conditions:

$$Y_t^{\mathrm{FP}} \approx \frac{E_t}{\left[\frac{7}{2}(F_t+F_b)\right]^{2/7}[7F_t]^{5/7}}, \tag{36}$$

$$Y_b^{\mathrm{FP}} \approx \frac{E_b}{\left[\frac{7}{2}(F_t+F_b)\right]^{2/7}[7F_b]^{2/7}\left[\frac{7}{3}(3F_b+F_\tau)\right]^{3/7}}, \tag{37}$$

$$Y_\tau^{\mathrm{FP}} \approx \frac{E_\tau}{\left[\frac{21}{4}F_\tau\right]^{4/7}\left[\frac{7}{3}(3F_b+F_\tau)\right]^{3/7}}. \tag{38}$$

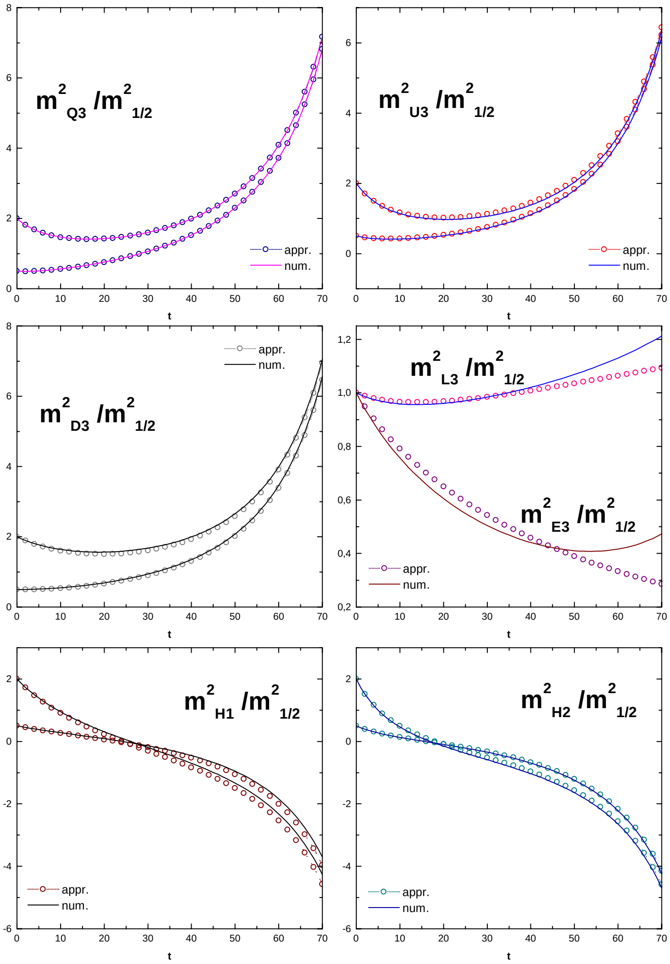

Fig. 3. Comparison of approximate and numerical solutions for the soft masses. The curves correspond to the following choice of initial conditions: $A_0 = 0$, $m_0^2 = (1/2)m_{1/2}^2$ and $m_0^2 = 2m_{1/2}^2$, $Y_{t0} = 5\alpha_0$, $Y_{b0} = Y_{\tau 0} = 2\alpha_0$. Slepton masses are shown only for $m_0^2 = m_{1/2}^2$. Dotted lines correspond to the analytical approximation, solid lines to the numerical solution

These expressions being expanded over the Grassmannian variables give the quasi-fixed points for the soft terms and masses. The explicit expressions are presented in Appendix B.

We see that the IRQFP behaviour is clearly expressed for Y_t and Y_b (see Fig. 4), and our approximate solution describes the fixed point line well. The same takes place for the corresponding As and Σs. For Y_τ, A_τ and Σ_τ the accuracy is worse, however, the solution is still reliable. The soft mass terms there exhibit the same IRQFP behaviour, though some residual dependence on the initial conditions is left in full analogy with the exact solutions in the low $\tan\beta$ case. The approximate solutions allow one to calculate the IRQFP analytically.

One can see that the fixed points for the soft terms naturally follow from the Grassmannian expansion of our approximate solutions (36)–(38), and they inherit their stability properties, as has been shown in [9]. In particular, the behaviour of the Σs essentially repeats that of the Yukawa couplings in agreement with [10].

The existence of the IRQFPs allows one to make predictions for the soft masses without exact knowledge of the initial conditions. This property has been widely used (see, for example, [11]) and though the IRQFPs give a slightly larger top mass when imposing b–τ unification; it is still possible to fit the quark masses within the error bars and to make predictions for the Higgs and sparticle spectrum [12]. This explains the general interest in the IRQFPs.

4 Discussion

We hope to convince the reader that the approximate solutions presented above reproduce the behaviour of the Yukawa couplings with good precision in the whole integration region and for a large range of initial values. The relative accuracy is typically a few per cent and is worse only at the end of the integration region mainly due to the smallness of the quantities themselves. Moreover, we have shown how the approximate solutions for the soft terms and masses follow from those for the rigid couplings. This demonstrates how the Grassmannian expansion, advocated in [3], works in the case of approximate solutions as well.

For illustration we have considered universal initial conditions for the soft terms. In recent time there appeared some interest in non-universal boundary conditions. Non-universality can also be included in our formulae at the expense of changing (10) and (34) using the same substitution rules, see (1) and (2).

Since the form of our approximate solutions has been "guessed" ad hoc starting from some reasonable arguments, there is no direct way to improve them. However, one can imagine a more constructive derivation of those solutions which would allow one to make corrections. Needless to say that it is enough to construct a solution for the rigid terms. Solutions for the soft terms will follow automatically.

Acknowledgements. We would like to thank A.V. Gladyshev for valuable discussions. Financial support from RFBR grants # 99-02-16650 and # 96-15-96030 is kindly acknowledged.

Appendix A

We here present approximate expressions for the soft couplings and masses corresponding to (31)–(33):

$$M_i = \frac{m_{1/2}}{1 + b_i\alpha_0 t},$$

$$A_t \approx A_0\left(1 - \frac{Y_{t0}F_t + Y_{b0}F_b}{1 + \frac{7}{2}(Y_{t0}F_t + Y_{b0}F_b)} - \frac{5Y_{t0}F_t}{1 + 7Y_{t0}F_t}\right)$$

$$-m_{1/2}\left(\frac{t}{E_t}\frac{\mathrm{d}E_t}{\mathrm{d}t} - \frac{Y_{t0}(tE_t - F_t) + Y_{b0}(tE_b - F_b)}{1 + \frac{7}{2}(Y_{t0}F_t + Y_{b0}F_b)}\right.$$

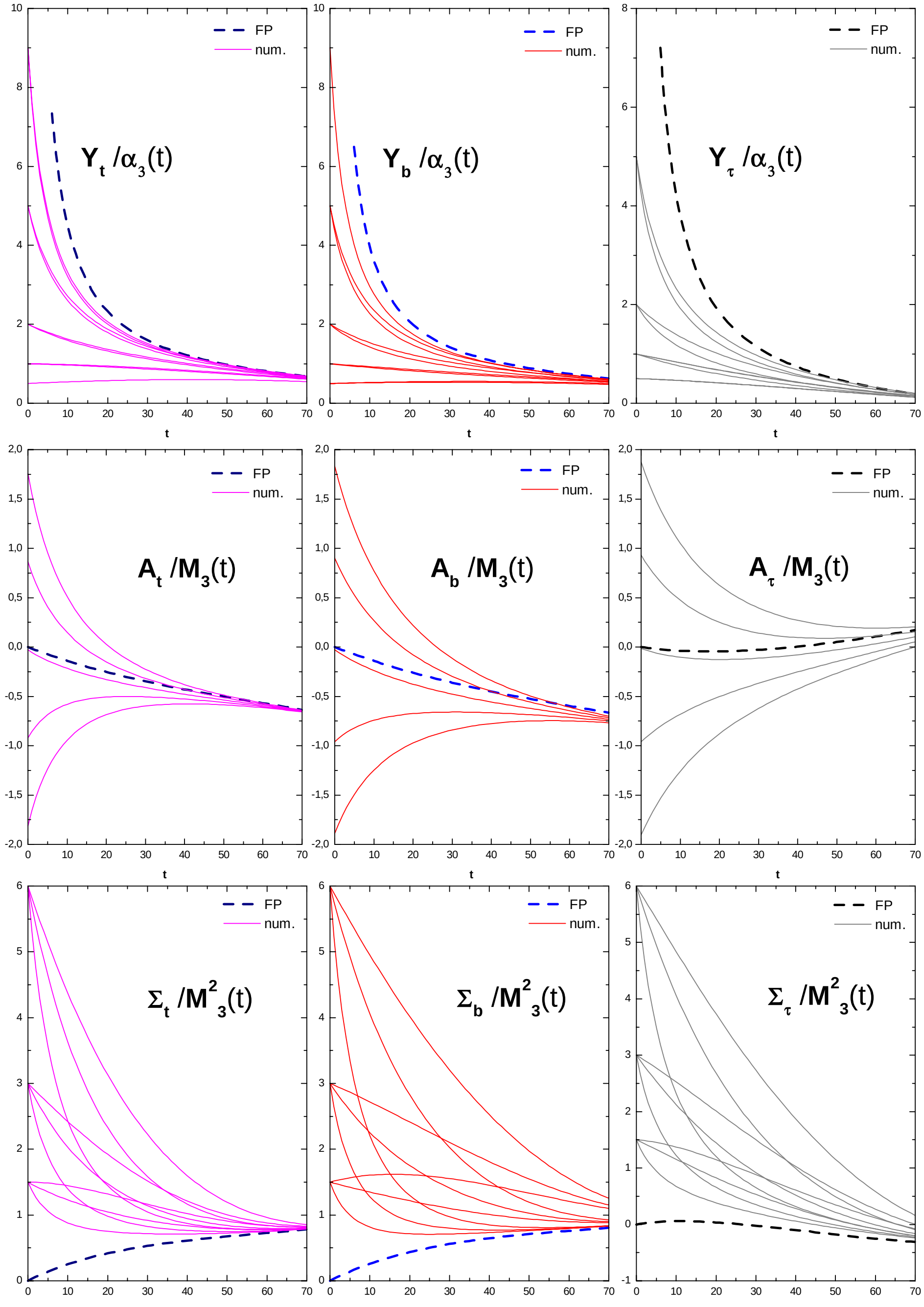

Fig. 4. The IRQFP behaviour for Yukawa couplings, soft terms $A_{t,b,\tau}$, and the Σs. Numerical solutions for all As are given for $Y_{t0}=5\alpha_0, Y_{b0}=Y_{\tau 0}=2\alpha_0$. For Σ we took $A_0=0$, three different sets of Y_0s and m_0^2. The dotted lines are the IRQFP from (36)–(38) and Appendix B

$$\left.-\frac{5Y_{t0}(tE_t-F_t)}{1+7Y_{t0}F_t}\right),$$

$$A_b \approx A_0\left(1-\frac{Y_{t0}F_t+Y_{b0}F_b}{1+\frac{7}{2}(Y_{t0}F_t+Y_{b0}F_b)}-\frac{2Y_{b0}F_b}{1+7Y_{b0}F_b}\right.$$

$$\left.-\frac{3Y_{b0}F_b+Y_{\tau 0}F_\tau}{1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau 0}F_\tau)}\right)$$

$$-m_{1/2}\left(\frac{t}{E_b}\frac{\mathrm{d}E_b}{\mathrm{d}t}-\frac{Y_{t0}(tE_t-F_t)+Y_{b0}(tE_b-F_b)}{1+\frac{7}{2}(Y_{t0}F_t+Y_{b0}F_b)}\right.$$

$$-\frac{2Y_{b0}(tE_b-F_b)}{1+7Y_{b0}F_b}$$

$$\left.-\frac{3Y_{b0}(tE_b-F_b)+Y_{\tau 0}(tE_\tau-F_\tau)}{1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau 0}F_\tau)}\right),$$

$$A_\tau \approx A_0\left(1-\frac{3Y_{\tau 0}F_\tau}{1+\frac{21}{4}Y_{\tau 0}F_\tau}-\frac{3Y_{b0}F_b+Y_{\tau 0}F_\tau}{1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau 0}F_\tau)}\right)$$

$$-m_{1/2}\left(\frac{t}{E_\tau}\frac{\mathrm{d}E_\tau}{\mathrm{d}t}-\frac{3Y_{\tau 0}(tE_\tau-F_\tau)}{1+\frac{21}{4}Y_{\tau 0}F_\tau}\right.$$

$$\left.-\frac{3Y_{b0}(tE_b-F_b)+Y_{\tau 0}(tE_\tau-F_\tau)}{1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau 0}F_\tau)}\right),$$

$$\Sigma_t \approx \Sigma_0\left(1-\frac{Y_{t0}F_t+Y_{b0}F_b}{1+\frac{7}{2}(Y_{t0}F_t+Y_{b0}F_b)}-\frac{5Y_{t0}F_t}{1+7Y_{t0}F_t}\right)$$

$$
\begin{aligned}
&-\frac{1}{1+\frac{7}{2}(Y_{t0}F_t+Y_{b0}F_b)}\\
&\times\Bigg(A_0^2Y_{t0}F_t-2A_0Y_{t0}m_{1/2}(tE_t-F_t)\\
&+Y_{t0}m_{1/2}^2t^2\frac{\mathrm{d}E_t}{\mathrm{d}t}+A_0^2Y_{b0}F_b-2A_0Y_{b0}m_{1/2}(tE_b-F_b)\\
&+Y_{b0}m_{1/2}^2t^2\frac{\mathrm{d}E_b}{\mathrm{d}t}\Bigg)\\
&-\frac{5}{1+7Y_{t0}F_t}\Bigg(A_0^2Y_{t0}F_t-2A_0Y_{t0}m_{1/2}(tE_t-F_t)\\
&+Y_{t0}m_{1/2}^2t^2\frac{\mathrm{d}E_t}{\mathrm{d}t}\Bigg)\\
&+\frac{\frac{7}{2}}{\left(1+\frac{7}{2}(Y_{t0}F_t+Y_{b0}F_b)\right)^2}\Big(-A_0Y_{t0}F_t\\
&+Y_{t0}m_{1/2}(tE_t-F_t)-A_0Y_{b0}F_b\\
&+Y_{b0}m_{1/2}(tE_b-F_b)\Big)^2+m_{1/2}^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{E_t}\frac{\mathrm{d}E_t}{\mathrm{d}t}\right)\\
&+\frac{35}{\left(1+7Y_{t0}F_t\right)^2}\Big(-A_0Y_{t0}F_t+Y_{t0}m_{1/2}(tE_t-F_t)\Big)^2,
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_b\approx\Sigma_0&\Bigg(1-\frac{Y_{t0}F_t+Y_{b0}F_b}{1+\frac{7}{2}(Y_{t0}F_t+Y_{b0}F_b)}-\frac{2Y_{b0}F_b}{1+7Y_{b0}F_b}\\
&-\frac{3Y_{b0}F_b+Y_{\tau0}F_\tau}{1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau0}F_\tau)}\Bigg)\\
&-\frac{1}{1+\frac{7}{2}(Y_{t0}F_t+Y_{b0}F_b)}\\
&\times\Bigg(A_0^2Y_{t0}F_t-2A_0Y_{t0}m_{1/2}(tE_t-F_t)\\
&+Y_{t0}m_{1/2}^2t^2\frac{\mathrm{d}E_t}{\mathrm{d}t}+A_0^2Y_{b0}F_b\\
&-2A_0Y_{b0}m_{1/2}(tE_b-F_b)+Y_{b0}m_{1/2}^2t^2\frac{\mathrm{d}E_b}{\mathrm{d}t}\Bigg)\\
&-\frac{2}{1+7Y_{b0}F_b}\Bigg(A_0^2Y_{b0}F_b\\
&-2A_0Y_{b0}m_{1/2}(tE_b-F_b)+Y_{b0}m_{1/2}^2t^2\frac{dE_b}{dt}\Bigg)\\
&-\frac{1}{1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau0}F_\tau)}\Bigg(3A_0^2Y_{b0}F_b+A_0^2Y_{\tau0}F_\tau\\
&-6A_0Y_{b0}m_{1/2}(tE_b-F_b)-2A_0Y_{\tau0}m_{1/2}(tE_\tau-F_\tau)\\
&+3Y_{b0}m_{1/2}^2t^2\frac{\mathrm{d}E_b}{\mathrm{d}t}+Y_{\tau0}m_{1/2}^2t^2\frac{\mathrm{d}E_\tau}{\mathrm{d}t}\Bigg)\\
&+\frac{\frac{7}{2}}{\left(1+\frac{7}{2}(Y_{t0}F_t+Y_{b0}F_b)\right)^2}\Big(-A_0Y_{t0}F_t\\
&+Y_{t0}m_{1/2}(tE_t-F_t)-A_0Y_{b0}F_b\\
&+Y_{b0}m_{1/2}(tE_b-F_b)\Big)^2\\
&+m_{1/2}^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{E_b}\frac{\mathrm{d}E_b}{\mathrm{d}t}\right)\\
&+14\frac{\left(-A_0Y_{b0}F_b+Y_{b0}m_{1/2}(tE_b-F_b)\right)^2}{\left(1+7Y_{b0}F_b\right)^2}\\
&+\frac{\frac{7}{3}}{\left(1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau0}F_\tau)\right)^2}\Big(-3A_0Y_{b0}F_b-A_0Y_{\tau0}F_\tau\\
&+3Y_{b0}m_{1/2}(tE_b-F_b)+Y_{\tau0}m_{1/2}(tE_\tau-F_\tau)\Big)^2,
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_\tau\approx\Sigma_0&\left(1-\frac{3Y_{\tau0}F_\tau}{1+\frac{21}{4}Y_{\tau0}F_\tau}-\frac{3Y_{b0}F_b+Y_{\tau0}F_\tau}{1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau0}F_\tau)}\right)\\
&-\frac{3}{1+\frac{21}{4}Y_{\tau0}F_\tau}\Big(A_0^2Y_{\tau0}F_\tau-2A_0Y_{\tau0}m_{1/2}(tE_\tau-F_\tau)\\
&+Y_{\tau0}m_{1/2}^2t^2\frac{\mathrm{d}E_\tau}{\mathrm{d}t}\Big)\\
&-\frac{1}{1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau0}F_\tau)}\Big(3A_0^2Y_{b0}F_b+A_0^2Y_{\tau0}F_\tau\\
&-6A_0Y_{b0}m_{1/2}(tE_b-F_b)+3Y_{b0}m_{1/2}^2t^2\frac{\mathrm{d}E_b}{\mathrm{d}t}\\
&-2A_0Y_{\tau0}m_{1/2}(tE_\tau-F_\tau)+Y_{\tau0}m_{1/2}^2t^2\frac{\mathrm{d}E_\tau}{\mathrm{d}t}\Big)\\
&+\frac{63}{4}\frac{\left(-A_0Y_{\tau0}F_\tau+Y_{\tau0}m_{1/2}(tE_\tau-F_\tau)\right)^2}{\left(1+\frac{21}{4}Y_{\tau0}F_\tau\right)^2}\\
&+\frac{\frac{7}{3}}{\left(1+\frac{7}{3}(3Y_{b0}F_b+Y_{\tau0}F_\tau)\right)^2}\big(-3A_0Y_{b0}F_b-A_0Y_{\tau0}F_\tau\\
&+3Y_{b0}m_{1/2}(tE_b-F_b)+Y_{\tau0}m_{1/2}(tE_\tau-F_\tau)\big)^2\\
&+m_{1/2}^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{E_\tau}\frac{\mathrm{d}E_\tau}{\mathrm{d}t}\right).
\end{aligned}
$$

To find the individual soft masses one can formally perform integration of the RG equations and express the masses through Σs solving a system of linear algebraic equations. This gives

$$
\begin{aligned}
m_{Q_3}^2&=\frac{13}{61}m_0^2+m_{1/2}^2\left(\frac{64}{61}f_3+\frac{87}{122}f_2-\frac{11}{122}f_1\right)\\
&+\frac{1}{122}\left(17\Sigma_t+20\Sigma_b-5\Sigma_\tau\right), \qquad (39)\\
m_{U_3}^2&=\frac{7}{61}m_0^2+m_{1/2}^2\left(\frac{72}{61}f_3-\frac{54}{61}f_2+\frac{72}{305}f_1\right)\\
&+\frac{1}{122}\left(42\Sigma_t-8\Sigma_b+2\Sigma_\tau\right), \qquad (40)\\
m_{D_3}^2&=\frac{19}{61}m_0^2+m_{1/2}^2\left(\frac{56}{61}f_3-\frac{42}{61}f_2+\frac{56}{305}f_1\right)\\
&+\frac{1}{122}\left(-8\Sigma_t+48\Sigma_b-12\Sigma_\tau\right), \qquad (41)\\
m_{H_1}^2&=-\frac{32}{61}m_0^2+m_{1/2}^2\left(-\frac{120}{61}f_3-\frac{3}{122}f_2-\frac{57}{610}f_1\right)\\
&+\frac{1}{122}\left(-9\Sigma_t+54\Sigma_b+17\Sigma_\tau\right), \qquad (42)\\
m_{H_2}^2&=-\frac{20}{61}m_0^2+m_{1/2}^2\left(-\frac{136}{61}f_3+\frac{21}{122}f_2-\frac{89}{610}f_1\right)
\end{aligned}
$$

$$+\frac{1}{122}(63\Sigma_t - 12\Sigma_b + 3\Sigma_\tau), \quad (43)$$

$$m^2_{L_3} = \frac{31}{61}m_0^2 + m^2_{1/2}\left(\frac{40}{61}f_3 + \frac{123}{122}f_2 - \frac{103}{610}f_1\right) + \frac{1}{122}(3\Sigma_t - 18\Sigma_b + 35\Sigma_\tau), \quad (44)$$

$$m^2_{E_3} = \frac{1}{61}m_0^2 + m^2_{1/2}\left(\frac{80}{61}f_3 - \frac{60}{61}f_2 + \frac{16}{61}f_1\right) + \frac{1}{122}(6\Sigma_t - 36\Sigma_b + 70\Sigma_\tau). \quad (45)$$

The masses of the squarks and sleptons of the first two generations are given by (19)–(14).

Appendix B

We here present the IRQFPs for the soft couplings and masses. They are obtained via Grassmannian expansion of (36)–(38).

$$A_t^{\rm FP} \approx -m_{1/2}\left(\frac{t}{E_t}\frac{{\rm d}E_t}{{\rm d}t} - 2\frac{(tE_t - F_t) + (tE_b - F_b)}{7(F_t+F_b)} - \frac{5(tE_t - F_t)}{7F_t}\right),$$

$$A_b^{\rm FP} \approx -m_{1/2}\left(\frac{t}{E_b}\frac{{\rm d}E_b}{{\rm d}t} - 2\frac{(tE_t - F_t) + (tE_b - F_b)}{7(F_t+F_b)} - \frac{2(tE_b - F_b)}{7F_b} - 3\frac{3(tE_b - F_b) + (tE_\tau - F_\tau)}{7(3F_b+F_\tau)}\right),$$

$$A_\tau^{\rm FP} \approx -m_{1/2}\left(\frac{t}{E_\tau}\frac{{\rm d}E_\tau}{{\rm d}t} - \frac{4(tE_\tau - F_\tau)}{7F_\tau} - 3\frac{3(tE_b - F_b) + (tE_\tau - F_\tau)}{7(3F_b+F_\tau)}\right),$$

$$\Sigma_t^{\rm FP} \approx m^2_{1/2}\left(-\frac{2}{7}\frac{t^2\frac{{\rm d}E_t}{{\rm d}t} + t^2\frac{{\rm d}E_b}{{\rm d}t}}{(F_t+F_b)} - \frac{5}{7}\frac{t^2\frac{{\rm d}E_t}{{\rm d}t}}{F_t} + \frac{\rm d}{{\rm d}t}\left(\frac{t^2}{E_t}\frac{{\rm d}E_t}{{\rm d}t}\right) + \frac{2}{7}\frac{[(tE_t - F_t) + (tE_b - F_b)]^2}{(F_t+F_b)^2} + \frac{5}{7}\frac{(tE_t - F_t)^2}{F_t^2}\right),$$

$$\Sigma_b^{\rm FP} \approx m^2_{1/2}\left(-\frac{2}{7}\frac{t^2\frac{{\rm d}E_t}{{\rm d}t} + t^2\frac{{\rm d}E_b}{{\rm d}t}}{(F_t+F_b)} - \frac{2}{7}\frac{t^2\frac{{\rm d}E_b}{{\rm d}t}}{F_b} - \frac{3}{7}\frac{3t^2\frac{{\rm d}E_b}{{\rm d}t} + t^2\frac{{\rm d}E_\tau}{{\rm d}t}}{(3F_b+F_\tau)} + \frac{2}{7}\frac{[(tE_t - F_t) + (tE_b - F_b)]^2}{(F_t+F_b)^2} + \frac{2}{7}\frac{(tE_b - F_b)^2}{F_b^2} + \frac{3}{7}\frac{[3(tE_b - F_b) + (tE_\tau - F_\tau)]^2}{(3F_b+F_\tau)^2} + \frac{\rm d}{{\rm d}t}\left(\frac{t^2}{E_b}\frac{{\rm d}E_b}{{\rm d}t}\right)\right),$$

$$\Sigma_\tau^{\rm FP} \approx m^2_{1/2}\left(-\frac{4}{7}\frac{t^2\frac{{\rm d}E_\tau}{{\rm d}t}}{F_\tau} - \frac{3}{7}\frac{3t^2\frac{{\rm d}E_b}{{\rm d}t} + t^2\frac{{\rm d}E_\tau}{{\rm d}t}}{(3F_b+F_\tau)} + \frac{\rm d}{{\rm d}t}\left(\frac{t^2}{E_\tau}\frac{{\rm d}E_\tau}{{\rm d}t}\right) + \frac{4}{7}\frac{(tE_\tau - F_\tau)^2}{F_\tau^2} + \frac{3}{7}\frac{[3(tE_b - F_b) + (tE_\tau - F_\tau)]^2}{(3F_b+F_\tau)^2}\right).$$

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